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Non-Abelian instantons on a fuzzy four-sphere

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Abstract

We study the compatibility between the BPST $SU(2)$ instanton and the fuzzy four-sphere algebra. By using the projective module point of view as an intermediate step, we are able to identify a noncommutative solution of the matrix model equations of motion which minimally extends the $SU(2)$ instanton solution on the classical sphere S^4. We also propose to extend the non-trivial second Chern class with the five-dimensional noncommutative Chern–Simons term.

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1. Introduction

Among all the known noncommutative varieties, the fuzzy four-sphere deformation has attracted our attention because it is the only one that incorporates the Kaluza–Klein mechanism in an elegant and mathematically clean way [1].

As an explicit application, we have devoted this work to the study of the topologically non-trivial configurations on a fuzzy four-sphere, having as a classical limit the $SU(2)$ BPST instantons on S^4.

The presence of the extra coordinates $(w_{\mu\nu})$ complicates the classical limit and requires selecting between those topologically non-trivial configurations admitting $w_{\mu\nu}$—decoupling from those for which this is not possible.

The projective module point of view clarifies the whole picture [2–8]. In fact at the $U(1)$ level, we can conceive noncommutative projectors [5] associated, through a simple link with matrix models [1, 9–11], with reducible representations of the fuzzy four-sphere algebra, but in this case the $w_{\mu\nu}$ decoupling is impossible because there is no analogous projector in the commutative limit [12, 13].

That is why we have concentrated our attention, in this work, to the non-Abelian case, where some classical projectors exist [2] (describing the BPST instantons of the gauge group $SU(2)$) on which the noncommutative deformation can be based.

We show that it is possible to deform the classical instantons, allowing a dependence on the extra coordinates $(w_{\mu\nu})$, weighted by a noncommutative damping factor $\rho \simeq \frac{1}{N}$, which explains their decoupling in the $N \to \infty$.

With a careful analysis of the link between projectors and matrix models, it is possible to identify which connections do admit as a classical limit the BPST instantons. They have a structure similar to the 't Hooft–Polyakov monopoles connections on a fuzzy sphere; see [15]. Moreover the 4d analysis is quite similar to the 2d case, apart from a fundamental passage, requiring the specific structure of the Hopf fibration $\pi : S^7 \to S^4$, which is an intrinsic 4d property.

It is also necessary extending the gauge group from $U(2)$ to $U(4)$ because of the dimensionality of the fuzzy four-sphere representations. In the last part of this work, we suggest how to build a candidate for extending the second Chern class, the only non-trivial one at a classical level.

It is convenient to take the 5d Chern–Simons term and we find perfect agreement with the classical limit, if we compare the noncommutative connection with the background of a $U(2)$ gauge theory (instead of $U(4)$). These last results are still 'under construction' since there is a lack of a deeper mathematical understanding of the intrinsic topological meaning of such configurations.

2. Review of the fuzzy four-sphere

The fuzzy four-sphere is a noncommutative manifold, defined by the following two general conditions

$$\epsilon^{\mu\nu\lambda\rho\sigma}\hat{x}_\mu\hat{x}_\nu\hat{x}_\lambda\hat{x}_\rho = C\hat{x}_\sigma \qquad \hat{x}_\mu\hat{x}_\mu = R^2 \tag{2.1}$$

where R is the radius of the sphere. One of these two constraints becomes trivial in the classical limit. Both conditions can be solved by introducing some auxiliary finite matrices $\hat{G}_\mu$ as follows,

$$\hat{x}_\mu = \rho\hat{G}_\mu \tag{2.2}$$

where $\hat{G}_\mu$ is the N-fold symmetrized tensorial product of the gamma matrices; see [1] for details.

The irreducible representations of $\hat{G}_\mu$ are labelled by this number N, from which the dimension d_N of the representation $\tilde{G}_\mu^{(N)}$ is determined to be

$$d_N = \frac{(N+1)(N+2)(N+3)}{6}. \tag{2.3}$$

The C constant depends of course on the integer N. By adding the auxiliary matrices $\hat{G}_{\mu\nu}^{(N)} = \frac{1}{2}[\hat{G}_\mu, \hat{G}_\nu]$, we can define a closed algebra $(SO(5,1))$, with the following commutation rules:

$$\begin{aligned}\left[\hat{G}_\mu^{(N)}, \hat{G}_{\nu\lambda}^{(N)}\right] &= 2\left(\delta_{\mu\nu}\hat{G}_\lambda^{(N)} - \delta_{\mu\lambda}\hat{G}_\nu^{(N)}\right)\\ \left[\hat{G}_{\mu\nu}^{(N)}, \hat{G}_{\lambda\rho}^{(N)}\right] &= 2\left(\delta_{\nu\lambda}\hat{G}_{\mu\rho}^{(N)} + \delta_{\mu\rho}\hat{G}_{\nu\lambda}^{(N)} - \delta_{\mu\lambda}\hat{G}_{\nu\rho}^{(N)} - \delta_{\nu\rho}\hat{G}_{\mu\lambda}^{(N)}\right).\end{aligned} \tag{2.4}$$

This means in practice that we must extend the set of basic coordinates of the four-sphere from five to fifteen:

$$\hat{x}_\mu = \rho\hat{G}_\mu^{(N)} \qquad \hat{w}_{\mu\nu} = \mathrm{i}\rho\hat{G}_{\mu\nu}^{(N)} = \frac{\mathrm{i}\rho}{2}\left[\hat{G}_\mu^{(N)}, \hat{G}_\nu^{(N)}\right]. \tag{2.5}$$

On the fuzzy four-sphere, the noncommutative coordinates are constrained to be

$$\begin{aligned}&[\hat{x}_\mu, \hat{x}_\nu] = -2\mathrm{i}\rho\hat{w}_{\mu\nu}\\ &\epsilon^{\mu\nu\lambda\rho\sigma}\hat{w}_{\mu\nu}\hat{w}_{\lambda\rho} = -\rho(8N+16)\hat{x}_\sigma.\end{aligned} \tag{2.6}$$

The classical sphere S^4 is reobtained from the large-N limit keeping fixed the radius of the sphere R ($\rho \to 0$). Under such a limit, all these coordinates become commutative, but this does not mean that the extra coordinates $w_{\mu\nu}$ are decoupled from the dynamics. Such a property must be imposed as a dynamical request (see later the discussion of the classical limit of the instantons).

The algebra (2.4) satisfies several interesting relations that can be summarized as follows:

$$\begin{aligned} &\hat{G}^{(N)}_\mu \hat{G}^{(N)}_\mu = N(N+4) = c \\ &\epsilon^{\mu\nu\lambda\rho\sigma} \hat{G}^{(N)}_\mu \hat{G}^{(N)}_\nu \hat{G}^{(N)}_\lambda \hat{G}^{(N)}_\rho = \epsilon^{\mu\nu\lambda\rho\sigma} \hat{G}^{(N)}_{\mu\nu} \hat{G}^{(N)}_{\lambda\rho} = (8N+16)\hat{G}^{(N)}_\sigma \end{aligned} \tag{2.7}$$

from which we can fix the constant C and the parameter ρ as

$$C = (8N+16)\rho^3 \qquad R^2 = \rho^2 N(N+4). \tag{2.8}$$

We can also add the following relations:

$$\begin{aligned} &\hat{G}^{(N)}_{\mu\nu} \hat{G}^{(N)}_\nu = 4\hat{G}^{(N)}_\mu \\ &\hat{G}^{(N)}_{\mu\nu} \hat{G}^{(N)}_{\nu\mu} = 4N(N+4) = 4c \\ &\hat{G}^{(N)}_{\mu\nu} \hat{G}^{(N)}_{\nu\lambda} = c\delta_{\mu\lambda} + \hat{G}^{(N)}_\mu \hat{G}^{(N)}_\lambda - 2\hat{G}^{(N)}_\lambda \hat{G}^{(N)}_\mu. \end{aligned} \tag{2.9}$$

The presence of the extra coordinates $\hat{w}_{\mu\nu}$ can be understood by asserting that a fuzzy two-sphere is attached to every point of the fuzzy four-sphere. In fact, we can take a representation of the algebra (2.4) in which the matrix $\hat{x}_5 = \rho\hat{G}_5$ is diagonal. Then it exists a subalgebra $SU(2) \times SU(2)$ generated by $\hat{G}_{\mu\nu}(\mu, \nu = 1, \ldots, 4)$ of the full $SO(5)$ algebra that commutes with $\hat{x}_5$. This subalgebra can be put in a diagonal form as follows:

$$[\hat{N}_i, \hat{N}_j] = \mathrm{i}\epsilon_{ijk}\hat{N}_k \qquad [\hat{M}_i, \hat{M}_j] = \mathrm{i}\epsilon_{ijk}\hat{M}_k \qquad [\hat{N}_i, \hat{M}_j] = 0 \tag{2.10}$$

where $\hat{N}_i$ and $\hat{M}_i$ are appropriate combinations of $\hat{G}_{\mu\nu}(\mu, \nu = 1, \ldots, 4)$; see [1]. The Casimir of each $SU(2)$ algebra can be computed in terms of the eigenvalue G_5 of the matrix $\hat{x}_5$:

$$\hat{N}_i\hat{N}_i = \tfrac{1}{16}(N+G_5)(N+4+G_5) \qquad \hat{M}_i\hat{M}_i = \tfrac{1}{16}(N-G_5)(N+4-G_5). \tag{2.11}$$

When the eigenvalue G_5 takes its maximum value N, i.e. we stay on top of the fuzzy four-sphere, one $SU(2)$ algebra decouples:

$$\hat{N}_i\hat{N}_i = \frac{N(N+2)}{4} \qquad \hat{M}_i\hat{M}_i = 0 \tag{2.12}$$

and therefore we can conclude that there is only one fuzzy two-sphere attached to the north pole of the fuzzy four-sphere. By using the $SO(5)$ symmetry we can extend this result to every point of the fuzzy four-sphere.

Such extra fuzzy two-sphere is a kind of internal space or spin, which of course complicates the analysis of a field theory defined on a fuzzy four-sphere.

Gauge theories can be defined by considering the following matrix model,

$$S = -\frac{1}{\rho^2}\mathrm{Tr}\left[\frac{1}{4}[X_\mu, X_\nu][X_\mu, X_\nu] + \frac{k}{5}\epsilon^{\mu\nu\lambda\rho\sigma} X_\mu X_\nu X_\lambda X_\rho X_\sigma\right] \tag{2.13}$$

where the indices $\mu, \nu, \ldots, \sigma$ take the values $1, \ldots, 5$ and are contracted with the Euclidean metric, while $\epsilon^{\mu\nu\rho\lambda\sigma}$ is the ($SO(5)$ invariant) totally antisymmetric tensor. X_μ are $d_N \times d_N$ Hermitian matrices (when d_N is defined in equation (2.3)), and k is a dimensional constant depending on N. The second term is known as Myers term and it can be understood in terms of branes [11].

The gauge symmetry is defined through the following unitary symmetry:

$$X_\mu = U^\dagger X_\mu U \qquad UU^\dagger = U^\dagger U = 1 \tag{2.14}$$

The k constant is determined by the condition that the matrix model admits as a classical solution the fuzzy four-sphere (2.1):

$$
\begin{aligned}
&X_\mu = \hat{x}_\mu = \rho \hat{G}_\mu^{(N)} \\
&[X_\nu, [X_\mu, X_\nu]] + k\epsilon^{\mu\nu\lambda\rho\sigma} X_\nu X_\lambda X_\rho X_\sigma = 0.
\end{aligned}
\tag{2.15}
$$

Then k can be identified with

$$
k = \frac{2}{\rho(N+2)}. \tag{2.16}
$$

It is also possible to define an action of Yang–Mills with mass term having the fuzzy four-sphere as classical solution:

$$
S = -\frac{1}{g^2}\mathrm{Tr}\left[\frac{1}{4}[X_\mu, X_\nu][X_\mu, X_\nu] + 8\rho^2 X_\mu X_\mu\right]. \tag{2.17}
$$

The more general action can be identified as a linear combination of two basic actions (2.13) and (2.17)

$$
\begin{aligned}
S(\lambda) = -\frac{1}{\rho^2}\mathrm{Tr}\bigg[&\frac{1}{4}[X_\mu, X_\nu][X_\mu, X_\nu] \\
&+ \frac{2\lambda}{5(N+2)\rho}\epsilon^{\mu\nu\lambda\rho\sigma} X_\mu X_\nu X_\lambda X_\rho X_\sigma + 8(1-\lambda)\rho^2 X_\mu X_\mu\bigg].
\end{aligned}
\tag{2.18}
$$

The term proportional to λ is the analogue of the five-dimensional Chern–Simons term which will be useful in the final discussion as a candidate for extending the second Chern class.

The construction of a noncommutative gauge theory on a fuzzy four-sphere is completed by developing the generic Hermitian matrices X_μ around the classical background $\hat{x}_\mu$:

$$
X_\mu = \hat{x}_\mu + \rho R \hat{a}_\mu. \tag{2.19}
$$

Since we are in a sort of Kaluza–Klein theory, when we develop the $d_N \times d_N$ fluctuation matrix $\hat{a}_\mu$, we must take into account that the field depends not only on the coordinates of the sphere, but also on the extra coordinates $\hat{w}_{\mu\nu}$. This requires extension of the basis of the functional space from the usual spherical harmonics to irreducible representations of $SO(5)$, which are labelled by two parameters r_1, r_2 with $0 \leqslant r_2 \leqslant r_1$. The condition to be on a fuzzy four-sphere implies a truncation of this functional space, which requires that the principal parameter r_1 is constrained to be $r_1 \leqslant N$ (for higher dimensional fuzzy spheres see for example [14]).

In the particular case $r_2 = 0$, the spherical harmonics of the sphere are recovered. Summarizing the general development of the fluctuation matrix $\hat{a}$ is given by

$$
\hat{a}(\hat{x}, \hat{w}) = \sum_{r_1=0}^{n}\sum_{r_2=0}^{r_1}\sum_{m_i} a_{r_1 r_2 m_i} \hat{Y}_{r_1 r_2 m_i}(\hat{x}, \hat{w}) \tag{2.20}
$$

where m_i are the extra quantum numbers which are necessary to label the representation.

We also need the definition of derivative operators

$$
\begin{aligned}
&Ad(\hat{G}_\mu) \to -2\mathrm{i}\left(w_{\mu\nu}\frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial w_{\mu\nu}}\right) \\
&Ad(\hat{G}_{\mu\nu}) \to 2\left(x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu} - w_{\mu\lambda}\frac{\partial}{\partial w_{\lambda\nu}} + w_{\nu\lambda}\frac{\partial}{\partial w_{\lambda\mu}}\right).
\end{aligned}
\tag{2.21}
$$

The integration of the classical gauge theory is replaced by the trace in the matrix model action. This correspondence is complicated by the presence of the two-dimensional internal space N_i.

Finally the Laplacian on the sphere has two possible extension, $\mathrm{Ad}(\hat{G}_{\mu\nu})^2$ and $\mathrm{Ad}(\hat{G}_{\mu})^2$. The natural choice for a matrix model, in which we develop the X_μ matrices around the background $\hat{x}_\mu = \rho\hat{G}_\mu$, is given by $\mathrm{Ad}(\hat{G}_\mu)^2$, that in the $w_{\mu\nu} \to 0$ limit reproduces the usual Laplacian on the classical four-sphere. The action of these two Laplacians on the spherical harmonics is as follows:

$$\begin{aligned} \tfrac{1}{4}\big[\hat{G}_\mu, \big[\hat{G}_\mu, \hat{Y}_{r_1,r_2}\big]\big] &= (r_1(r_1+3) - r_2(r_2+1))\hat{Y}_{r_1,r_2} \\ -\tfrac{1}{8}\big[\hat{G}_{\mu\nu}, \big[\hat{G}_{\mu\nu}, \hat{Y}_{r_1,r_2}\big]\big] &= (r_1(r_1+3) + r_2(r_2+1))\hat{Y}_{r_1,r_2}. \end{aligned} \tag{2.22}$$

3. $U(2)$ projectors and BPST instantons

In our article [5] we postulated some $U(1)$ noncommutative projectors, but then we realized that in this case the extra coordinates do not decouple in the classical limit.

Hence to reach a physically meaningful result it is necessary to study at a noncommutative level the case of $SU(2)$ instantons, classically described in terms of projective modules in [2].

We recall that to obtain a projective module description for an $SU(2)$ gauge theory on the S^4 sphere, it is necessary to make use of the Hopf projection $\pi : S^7 \to S^4$ and the quaternion field H. In particular, the function spaces we are interested in are $A_H = C^\infty(S^4, H)$, the algebra of smooth functions taking values in H on the basic space S^4, and $B_H = C^\infty(S^7, H)$, the algebra of smooth functions with values in H on the total space S^7.

The projector, whose elements belong to A_H, can be built, using the Hopf fibration $\pi : S^7 \to S^4$, in terms of a vector, whose elements belong to B_H, and this condition is important to assure the non-triviality of the projector and the intrinsic topological nature of the solution.

$$p = |\psi\rangle\langle\psi| \qquad \langle\psi|\psi\rangle = 1 \qquad |\psi\rangle = \begin{pmatrix} \psi_1 \\ \cdots \\ \cdots \\ \psi_N \end{pmatrix}. \tag{3.1}$$

To realize the Hopf projection $\pi : S^7 \to S^4$ it is necessary to introduce a couple of quaternions (analogously to the case $\pi : S^3 \to S^2$ which is described by a couple of complex coordinates) subject to the constraint

$$S^7 = \{(a, b) \in H^2, |a|^2 + |b|^2 = 1\} \tag{3.2}$$

on which the following right action is defined as

$$S^7 \times Sp(1) \to S^7 \qquad (a, b)w = (aw, bw) \qquad w \in Sp(1) \quad w\overline{w} = 1 \tag{3.3}$$

keeping invariant the S^7 sphere. In terms of the quaternions a, b the Hopf projection $\pi : S^7 \to S^4$ is realized as

$$\begin{gathered} x_1 = a\overline{b} + b\overline{a} \qquad \xi = a\overline{b} - b\overline{a} = -\overline{\xi} \qquad x_5 = |a|^2 - |b|^2 \\ \sum_{\mu=1}^{5} (x_\mu)^2 = (|a|^2 + |b|^2)^2 = 1. \end{gathered} \tag{3.4}$$

This mapping determines what are the $Sp(1)$ invariant combinations on S^7, i.e. functions taking values on S^4:

$$|a|^2 = \tfrac{1}{2}(1 + x_5) \qquad |b|^2 = \tfrac{1}{2}(1 - x_5) \qquad a\overline{b} = \tfrac{1}{2}(x_1 + \xi). \tag{3.5}$$

Then it is not difficult to construct, at least for instanton number $k = 1$ a projector, whose entries belong to the $Sp(1)$ invariant combinations, starting from the following vector $|\psi\rangle$:

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \tag{3.6}$$

satisfying the normalization condition $\langle\psi|\psi\rangle = |a|^2 + |b|^2 = 1$ on S^7. We can define a projector $p \in M_2(A_H)$ as

$$p = |\psi\rangle\langle\psi| = \begin{pmatrix} |a|^2 & a\overline{b} \\ b\overline{a} & |b|^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 + x_5 & x_1 + \xi \\ x_1 - \xi & 1 - x_5 \end{pmatrix}. \tag{3.7}$$

Clearly if we transform $|\psi\rangle$ with right action $Sp(1) : S^7 \times Sp(1) \to S^7$

$$|\psi\rangle \to |\psi^w\rangle = \begin{pmatrix} aw \\ bw \end{pmatrix} = |\psi\rangle w \qquad \forall w \in Sp(1) \tag{3.8}$$

the projector p remains invariant (another way to say that its elements belong to the A_H algebra instead of B_H).

The associated Chern classes are

$$C_1(p) = -\frac{1}{2\pi\mathrm{i}}\mathrm{Tr}(p(\mathrm{d}p)^2) \qquad C_2(p) = -\frac{1}{8\pi^2}[\mathrm{Tr}(p(\mathrm{d}p)^4) - (C_1(p))^2]. \tag{3.9}$$

Since the 2-form $p(\mathrm{d}p)^2$ has values in the pure imaginary quaternions, its trace is null

$$C_1(p) = 0. \tag{3.10}$$

Its second Chern class is instead non-trivial

$$C_2(p) = -\frac{3}{8\pi^2}\mathrm{d}(\mathrm{vol}(S^4)) \tag{3.11}$$

and it gives rise to a non-trivial topological number (Chern number)

$$c_2(p) = \int_{S^4} C_2(p) = -\frac{3}{8\pi^2}\int_{S^4} \mathrm{d}(\mathrm{vol}(S^4)) = -\frac{3}{8\pi^2}\frac{8\pi^2}{3} = -1. \tag{3.12}$$

The 1-form connection, associated with the projector p

$$A^\nabla = \langle\psi|\mathrm{d}|\psi\rangle = \overline{a}\,\mathrm{d}a + \overline{b}\,\mathrm{d}b \tag{3.13}$$

is anti-Hermitian taking values in the purely imaginary quaternions that can be identified with the Lie algebra $Sp(1) \sim SU(2)$.

The non-trivial moduli space of the $k = 1$ instanton is generated by the elements $g \in SL(2, H)$ (belonging to the conformal group), acting on the left:

$$|\psi\rangle \to |\psi^g\rangle = \frac{1}{[\langle\psi|g^\dagger g|\psi\rangle]^{\frac{1}{2}}} g|\psi\rangle. \tag{3.14}$$

The quotient of $SL(2, H)$ by the trivial subgroup $Sp(2)$ generates a five-parameter family of the $k = 1$ $SU(2)$ instantons.

At a noncommutative level, the existence of a classical projector helps in defining a corresponding noncommutative projector, whose necessary elements belong to the whole fuzzy four-sphere algebra, but where the dependence on the extra coordinates $\hat{w}_{\mu\nu}$ is controlled by a factor ρ $\left(\rho \simeq \frac{1}{N}\right)$. This assures the necessary decoupling of the extra coordinates in the $\rho \to 0$ limit, as physically required [13].

To reach such property we must rewrite the Hopf fibration $\pi : S^7 \to S^4$ in terms of four complex coordinates:

$$\begin{aligned}
&x_1 = \rho(\alpha_1 + \overline{\alpha}_1) \qquad x_2 = \mathrm{i}\rho(\alpha_1 - \overline{\alpha}_1)\\
&x_3 = \rho(\alpha_2 + \overline{\alpha}_2) \qquad x_4 = \mathrm{i}\rho(\alpha_2 - \overline{\alpha}_2)\\
&x_5 = \rho(a_0\overline{a}_0 + a_1\overline{a}_1 - a_2\overline{a}_2 - a_3\overline{a}_3)\\
&\alpha_1 = a_0\overline{a}_2 + a_3\overline{a}_1 \qquad \alpha_2 = a_0\overline{a}_3 - a_2\overline{a}_1\\
&\sum_i x_i^2 = \rho^2 \qquad \sum_i |a_i|^2 = 1.
\end{aligned} \tag{3.15}$$

In [5], we noted that quantizing such complex coordinates and keeping the same mapping (3.15)

$$\left[a_i, a_j^\dagger\right] = \delta_{ij} \qquad [a_i, a_j] = 0 \tag{3.16}$$

the resulting $\hat{x}_i$ coordinates are part of an algebra, coinciding with the fuzzy four-sphere algebra. In fact we can show that the total number operator

$$\hat{N} = a_0^\dagger a_0 + a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 \tag{3.17}$$

has an eigenvalue N on a irreducible representation of the algebra, and the Casimir $\hat{x}_i^2$ can be re-expressed in terms of the total number operator $\hat{N}$:

$$\sum_i \hat{x}_i^2 = \rho^2 \hat{N}(\hat{N} + 4) = R^2. \tag{3.18}$$

In terms of these oscillators, the noncommutative analogues of the quaternions (a, b) can be expressed as

$$a = \begin{pmatrix} a_0 & -a_1^\dagger \\ a_1 & a_0^\dagger \end{pmatrix} \qquad b = \begin{pmatrix} a_2 & -a_3^\dagger \\ a_3 & a_2^\dagger \end{pmatrix} \tag{3.19}$$

from which it is clear that the combination $a\overline{b}$ is inside the fuzzy four-sphere algebra, while $\overline{b}a$ is outside. The only combinations belonging to the algebra are of the type $(a\overline{a}, a\overline{b}, b\overline{a}, b\overline{b})$, which impose the following guess for the vector $|\psi_0\rangle$

$$|\psi_0\rangle = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{3.20}$$

By imposing the normalization condition

$$\langle\psi_0|\psi_0\rangle = (\overline{a}\overline{b})\begin{pmatrix} a \\ b \end{pmatrix} = \overline{a}a + \overline{b}b = \begin{pmatrix} \hat{N} & 0 \\ 0 & N \hat{+} 4 \end{pmatrix} \tag{3.21}$$

we obtain a diagonal matrix whose elements depend only on the total number operator $\hat{N}$. Redefining $|\psi_0\rangle$ as

$$|\psi_0\rangle \to |\psi\rangle = \begin{pmatrix} a' \\ b' \end{pmatrix} \qquad a' = a\sqrt{h(\hat{N})} \quad b' = b\sqrt{h(\hat{N})} \tag{3.22}$$

with the position

$$h(\hat{N}) = \begin{pmatrix} \frac{1}{\hat{N}} & 0 \\ 0 & \frac{1}{N \hat{+} 4} \end{pmatrix} \tag{3.23}$$

we can develop the resulting projector as

$$p_n = |\psi\rangle\langle\psi| = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} h(\hat{N}) & 0 \\ 0 & h(\hat{N}) \end{pmatrix} (\overline{a}\overline{b}) = \begin{pmatrix} ah(\hat{N})\overline{a} & ah(\hat{N})\overline{b} \\ bh(\hat{N})\overline{a} & bh(\hat{N})\overline{b} \end{pmatrix}. \tag{3.24}$$

We note that its elements are also functions of the extra coordinates $\hat{w}_{\mu\nu}$. However, differently from the $U(1)$ case, the dependence on the extra coordinates has a factor $\rho \sim \frac{1}{N}$ that assures their decoupling in the classical limit. We can write explicitly its first entry

$$
\begin{aligned}
ah(\hat{N})\overline{a} &= \begin{pmatrix} a_0 & -a_1^\dagger \\ a_1 & a_0^\dagger \end{pmatrix} \begin{pmatrix} \frac{1}{\hat{N}} & 0 \\ 0 & \frac{1}{\hat{N}+4} \end{pmatrix} \begin{pmatrix} a_0^\dagger & a_1^\dagger \\ -a_1 & a_0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\hat{N}+1} a_0 a_0^\dagger + \frac{1}{\hat{N}+3} a_1^\dagger a_1 & \left(\frac{1}{\hat{N}+1} - \frac{1}{\hat{N}+3}\right) a_0 a_1^\dagger \\ \left(\frac{1}{\hat{N}+1} - \frac{1}{\hat{N}+3}\right) a_1 a_0^\dagger & \frac{1}{\hat{N}+1} a_1 a_1^\dagger + \frac{1}{\hat{N}+3} a_0^\dagger a_0 \end{pmatrix}.
\end{aligned}
\tag{3.25}
$$

At this level we can substitute $\hat{N}$ with its eigenvalue N, making p_N a projector of the fuzzy four-sphere. The dependence on the extra coordinates is proportional to $\left(\frac{1}{N+1} - \frac{1}{N+3}\right)$, and therefore is vanishing for $N \to \infty$. The trace of the projector can be computed as

$$
\begin{aligned}
\mathrm{Tr} p_N &= \left(\frac{N+4}{N+1} + \frac{N}{N+3}\right) \mathrm{Tr} I \\
&= \left(\frac{(N+2)(N+3)(N+4)}{6} + \frac{N(N+1)(N+2)}{6}\right) \\
&= 2\,\mathrm{Tr}\, I + (N+2) < \mathrm{Tr}\, 1_P = 4\,\mathrm{Tr}\, I
\end{aligned}
\tag{3.26}
$$

and it is always an integer, for each value of N. There are no constraints on the possible values of N.

4. Connection with matrix models

As discussed in [15], the connection between projectors and matrix models in the non-Abelian case is not as simple as in the $U(1)$ case; however, it is still convenient starting from the following guess

$$
X_i = \langle \psi | G_i | \psi \rangle \tag{4.1}
$$

and correct this formula later on to reach a solution of the noncommutative equations of motion.

By looking at the form of $|\psi\rangle$

$$
|\psi\rangle = \begin{pmatrix} \begin{pmatrix} a_0 & -a_1^\dagger \\ a_1 & a_0^\dagger \end{pmatrix} \\ \begin{pmatrix} a_2 & -a_3^\dagger \\ a_3 & a_2^\dagger \end{pmatrix} \end{pmatrix} f(\hat{N}) \qquad f(\hat{N}) = \begin{pmatrix} \frac{1}{\sqrt{\hat{N}}} & 0 \\ 0 & \frac{1}{\sqrt{\hat{N}+4}} \end{pmatrix} \tag{4.2}
$$

it is not *a priori* obvious that the expectation value $\langle \psi | G_i | \psi \rangle$ reduces to a diagonal form. Since this really happens, we believe that such property is another consequence of the underlying Hopf projection $\pi : S^7 \to S^4$.

Let us check how it happens in detail. The expectation value

$$
\begin{aligned}
&\langle \psi | G_i | \psi \rangle \\
&\quad = \begin{pmatrix} \sum_k a_k^\dagger G_i a_k & -a_0^\dagger G_i a_1^\dagger + a_1^\dagger G_i a_0^\dagger - a_2^\dagger G_i a_3^\dagger + a_3^\dagger G_i a_2^\dagger \\ a_0 G_i a_1 - a_1 G_i a_0 + a_2 G_i a_3 - a_3 G_i a_2 & \sum_k a_k G_i a_k^\dagger \end{pmatrix} \\
&\qquad \cdot \begin{pmatrix} \frac{1}{\hat{N}} & 0 \\ 0 & \frac{1}{\hat{N}+4} \end{pmatrix}
\end{aligned}
\tag{4.3}
$$

contains in principle off-diagonal terms. For example let us check the term $(X_i)_{12}$

$$\begin{aligned}(X_i)_{12} &= -a_0^\dagger G_i a_1^\dagger + a_1^\dagger G_i a_0^\dagger - a_2^\dagger G_i a_3^\dagger + a_3^\dagger G_i a_2^\dagger \\ &= -a_0^\dagger\big[G_i, a_1^\dagger\big] + a_1^\dagger\big[G_i, a_0^\dagger\big] - a_2^\dagger\big[G_i, a_3^\dagger\big] + a_3^\dagger\big[G_i, a_2^\dagger\big].\end{aligned} \tag{4.4}$$

In the case $i = 5$ we obtain

$$\begin{aligned}&G_5 = a_0^\dagger a_0 + a_1^\dagger a_1 - a_2^\dagger a_2 - a_3^\dagger a_3 \\ &\quad\Rightarrow (X_5)_{12} = -a_0^\dagger a_1^\dagger + a_1^\dagger a_0^\dagger - a_2^\dagger a_3^\dagger + a_3^\dagger a_2^\dagger = 0.\end{aligned} \tag{4.5}$$

In the other cases we can simplify the discussion by taking the complex coordinates $G_{1\pm}$

$$\begin{aligned}&G_{1-} = a_0 a_2^\dagger + a_3 a_1^\dagger && G_{1+} = a_0^\dagger a_2 + a_3^\dagger a_1 \\ &\Rightarrow (X_{1-})_{12} = a_1^\dagger a_2^\dagger - a_2^\dagger a_1^\dagger = 0 && (X_{1+})_{12} = -a_0^\dagger a_3^\dagger + a_3^\dagger a_0^\dagger = 0\end{aligned} \tag{4.6}$$

and $G_{2\pm}$

$$\begin{aligned}&G_{2-} = a_0 a_3^\dagger - a_2 a_1^\dagger && G_{2+} = a_0^\dagger a_3 - a_2^\dagger a_1 \\ &\Rightarrow (X_{2-})_{12} = a_1^\dagger a_3^\dagger - a_3^\dagger a_1^\dagger = 0 && (X_{2+})_{12} = a_0^\dagger a_2^\dagger - a_2^\dagger a_0^\dagger = 0.\end{aligned} \tag{4.7}$$

In all cases a welcome cancellation appears, the reader can also verify it for the case $(X_i)_{21}$. The resulting diagonal terms can be computed by using the commutation relations of the oscillators

$$\langle\psi|G_i|\psi\rangle = \begin{pmatrix} \frac{\hat N - 1}{\hat N} G_i & 0 \\ 0 & \frac{\hat N + 5}{\hat N + 4} G_i \end{pmatrix}. \tag{4.8}$$

As we already discussed in our previous articles [12, 13], the G_i action on the vector $|\psi\rangle$ cannot be smoothly connected to ordinary derivative operators on the sphere in the classical limit unless we project the vector $|\psi\rangle$ on the fuzzy four-sphere algebra.

Since we must maintain invariant the form $U(2)$ of the projector (which is smoothly connected to the BPST $SU(2)$ instantons on the sphere S^4), we correct the vector $|\psi\rangle$ with a quasi-unitary operator such that $|\psi'\rangle = |\psi\rangle U$ has all elements belonging to the fuzzy four-sphere algebra:

$$|\psi'\rangle = |\psi\rangle U \quad UU^\dagger = 1 \qquad P = |\psi\rangle\langle\psi| = |\psi'\rangle\langle\psi'| \tag{4.9}$$

This operator U, as in the case of noncommutative monopoles, plays an essential role to define the non-Abelian topology. However, since the Hilbert space of the fuzzy four-sphere is generated by four oscillators, we find an unexpected difficulty in solving the constraint $UU^\dagger = 1$ with a 2×2 matrix.

This difficulty can be overcome only by embedding the $U(2)$ projector in a larger gauge group. In fact we can solve the constraint $UU^\dagger = 1$ with a 4×4 matrix, which requires embedding the projector in a $U(4)$ gauge theory.

During the discussion of the final solution, we will note that such difficulty is related to the dimensionality of the fuzzy four-sphere irreducible representations, defined in equation (2.3).

5. Embedding $U(2)$ in $U(4)$

In the case of $U(1)$ projectors (see [12, 13]), we were stuck into the problem of identifying the associated connections for a certain class of projectors (Serre–Swan theorem for the noncommutative case). In this section, the answer to this question comes out. The only obstacle, i.e. the construction of the quasi-unitary operator U, can always be overcome by embedding the projector into a larger gauge group.

In the present case solving the constraint $UU^\dagger = 1$, with U represented by 4×4 matrix is an easy task

$$U = \begin{pmatrix} U_{11}^\dagger & U_{12}^\dagger & U_{13}^\dagger & U_{14}^\dagger \\ 0 & U_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad U^\dagger = \begin{pmatrix} U_{11} & 0 & 0 & 0 \\ U_{12} & U_4^\dagger & 0 & 0 \\ U_{13} & 0 & 0 & 0 \\ U_{14} & 0 & 0 & 0 \end{pmatrix} \tag{5.1}$$

and the obvious solution can be represented as

$$\begin{aligned} U_{11} &= \sum_{n_1,n_2,n_3,n_4=0}^{\infty} |n_1, n_2, n_3, n_4\rangle\langle n_1 + 1, n_2, n_3, n_4| \\ U_{12} &= \sum_{n_2,n_3,n_4=0}^{\infty} |n_2, n_3, n_4, 0\rangle\langle 0, n_2 + 1, n_3, n_4| \\ U_{13} &= \sum_{n_3,n_4=0}^{\infty} |n_3, n_4, 0, 0\rangle\langle 0, 0, n_3 + 1, n_4| \\ U_{14} &= \sum_{n_4=0}^{\infty} |n_4, 0, 0, 0\rangle\langle 0, 0, 0, n_4 + 1| \\ U_4 &= \sum_{n_1,n_2,n_3,n_4=0}^{\infty} |n_1, n_2, n_3, n_4\rangle\langle n_1, n_2, n_3, n_4 + 1|. \end{aligned} \tag{5.2}$$

The structure of this solution is so unique that it is practically impossible to reduce it to a subspace of 2×2 matrices.

This quasi-unitary operator satisfies the following property:

$$UU^\dagger = \begin{pmatrix} \sum_{k=1}^4 U_{1k}^\dagger U_{1k} = 1 - |0,0,0,0\rangle\langle 0,0,0,0| & 0 & 0 & 0 \\ 0 & U_4 U_4^\dagger = 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.3}$$

$UU^\dagger$ is practically equivalent to the identity operator $1_{U(2)}$, apart from an operator $|0,0,0,0\rangle\langle 0,0,0,0|$, whose action is annihilated by the vectors $|\psi\rangle$.

For the combination $U^\dagger U$ we obtain, differently from the $2d$ case (see [15]):

$$U^\dagger U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sum_{n_3,n_4} |n_3, n_4, 0, 0\rangle\langle n_3, n_4, 0, 0| & 0 \\ 0 & 0 & 0 & \sum_{n_4} |n_4, 0, 0, 0\rangle\langle n_4, 0, 0, 0| \end{pmatrix}. \tag{5.4}$$

This property is strictly related to the dimensionality of the fuzzy four-sphere irreducible representations (see equation (2.3)).

The explicit construction of the quasi-unitary operator requires minimal extension of the vector $|\psi\rangle$, with entries belonging to $U(2)$ gauge group, to vectors with values in $U(4)$. Since the extension must be minimal, we restrict its non-trivial contribution to a $U(2)$ subgroup

$$|\psi\rangle = \begin{pmatrix} \begin{pmatrix} a_0 & -a_1^\dagger & 0 & 0 \\ a_1 & a_0^\dagger & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} a_2 & -a_3^\dagger & 0 & 0 \\ a_3 & a_2^\dagger & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{pmatrix} f(\hat{N}). \tag{5.5}$$

The expectation value (4.8) with this new vector $|\psi\rangle$ is similar to the $U(2)$ case:

$$X_i = \langle\psi|G_i|\psi\rangle = \begin{pmatrix} \frac{\hat{N}-1}{\hat{N}}G_i & 0 & 0 & 0 \\ 0 & \frac{\hat{N}+5}{\hat{N}+4}G_i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.6}$$

Correcting the vector $|\psi\rangle$ with 4×4 quasi-unitary operators changes the dimension of the background representation with a non-trivial mixing similar to the $2d$ case. Therefore we must expect that

$$\left(U^\dagger X_i^{bg} U\right)_{N+1} = \left(\begin{array}{c|c|c} (G_i)_{N+2} & 0 & 0 \\ \hline 0 & (G_i)_N & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \tag{5.7}$$

where the division in blocks is different in the last matrix. Such property is discussed in detail in the appendix, but we give now an indirect argument. As a consistency check of equation (5.7), we note that by taking the square of the background an interesting simplification appears. In fact since it is proportional to a Casimir of the fuzzy four-sphere algebra, its trace is a calculable number:

$$\begin{aligned} \mathrm{Tr}\left(X_i^{bg}\right)^2_{N+1} &= 4\,\mathrm{Tr}(G_i)_{N+1}(G_i)_{N+1} \\ &= 4(N+1)(N+5)\mathrm{Tr}1_{N+1} = \tfrac{2}{3}(N+1)(N+2)(N+3)(N+4)(N+5). \end{aligned} \tag{5.8}$$

Now let us take the square of equations (5.7) and compare the trace of both members. If we find agreement between these two numbers, we have an indirect proof that equation (5.7) is correct. Firstly we compute

$$\mathrm{Tr}\left(U^\dagger\left(X_i^{bg}X_i^{bg}\right)U\right)_{N+1} = \mathrm{Tr}\left(U_1^\dagger\left(X_i^{bg}X_i^{bg}\right)U_1\right)_{N+1} + \mathrm{Tr}\left(U_2^\dagger\left(X_i^{bg}X_i^{bg}\right)U_2\right)_{N+1} \tag{5.9}$$

where we have split the quasi-unitary operator U into the following two parts ($U = U_1 + U_2$):

$$U_1 = \begin{pmatrix} U_{11}^\dagger & U_{12}^\dagger & U_{13}^\dagger & U_{14}^\dagger \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad U_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & U_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.10}$$

This decomposition permits the following manipulations:

$$\mathrm{Tr}\left(U_1^\dagger\left(X_i^{bg}X_i^{bg}\right)U_1\right)_{N+1} = (N+2)(N+6)\mathrm{Tr}\left(U_1^\dagger U_1\right)$$
$$\mathrm{Tr}\left(U_2^\dagger\left(X_i^{bg}X_i^{bg}\right)U_2\right)_{N+1} = N(N+4)\mathrm{Tr}\left(U_2^\dagger U_2\right)$$

$$\mathrm{Tr}\big(U_1^\dagger U_1\big) = \sum_{k=1}^{4} \mathrm{Tr}\big(U_{1k}U_{1k}^\dagger\big) = \mathrm{Tr}(1)_{N+1} + (\mathrm{Tr}(1)_{N+1} - \mathrm{Tr}(1)_N) + (N+2) + 1 = \mathrm{Tr}(1)_{N+2}$$
$$\mathrm{Tr}\big(U_2^\dagger U_2\big) = \mathrm{Tr}\big(U_4^\dagger U_4\big) = \mathrm{Tr}(1)_N. \tag{5.11}$$

In summary we obtain

$$\mathrm{Tr}\big(U^\dagger \big(X_i^{bg} X_i^{bg}\big) U\big)_{N+1} = N(N+4)\mathrm{Tr}(1)_N + (N+2)(N+6)\mathrm{Tr}(1)_{N+2} \tag{5.12}$$

On the other hand, the second member of equation (5.7) gives rise to

$$\mathrm{Tr}\left(\begin{array}{c|c|c} (G_i)_{N+2} & 0 & 0 \\ \hline 0 & (G_i)_N & 0 \\ \hline 0 & 0 & 0 \end{array}\right)^2 = (G_i)_N(G_i)_N\mathrm{Tr}(1)_N + (G_i)_{N+2}(G_i)_{N+2}\mathrm{Tr}(1)_{N+2} \tag{5.13}$$

which is exactly the same number.

Proceeding this way, the expectation value for $|\psi'\rangle = |\psi\rangle U$ can be explicitly computed as

$$|\psi'\rangle = |\psi\rangle U \qquad P = |\psi'\rangle\langle\psi'| = |\psi\rangle\langle\psi|$$
$$X_i^{(0)} = \langle\psi'|X_i^{bg}|\psi'\rangle = \left(\begin{array}{c|c|c} \frac{\hat N}{\hat N+1}(G_i)_{N+2} & 0 & 0 \\ \hline 0 & \frac{\hat N+4}{\hat N+3}(G_i)_N & 0 \\ \hline 0 & 0 & 0 \end{array}\right). \tag{5.14}$$

This block diagonal matrix is still not an explicit solution of the noncommutative equations of motion. However as in the $2d$ case, the nearest solution is easy to reach, by redefining the relation between X_i and the vector $|\psi'\rangle$ in the following way:

$$X_i^{\mathrm{tot}} = \begin{pmatrix} f_+ & & & \\ & f_- & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \qquad X_i^{(0)} \begin{pmatrix} f_+ & & & \\ & f_- & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$
$$= \left(\begin{array}{c|c|c} (G_i)_{N+2} & 0 & 0 \\ \hline 0 & (G_i)_N & 0 \\ \hline 0 & 0 & 0 \end{array}\right). \tag{5.15}$$

The unknown constants $f_\pm$ are constrained to be

$$\begin{cases} (f_+ + f_-)^2 = \dfrac{N+1}{N} \\ (f_+ - f_-)^2 = \dfrac{N+1}{N+4} \end{cases} \qquad \begin{cases} f_+ = \dfrac{1}{2}\left(\sqrt{\dfrac{N+1}{N}} + \sqrt{\dfrac{N+1}{N+4}}\right) \\ f_- = \dfrac{1}{2}\left(\sqrt{\dfrac{N+1}{N}} - \sqrt{\dfrac{N+1}{N+4}}\right) \end{cases}. \tag{5.16}$$

In conclusion, the compatibility of the $SU(2)$ BPST instanton with the noncommutative structure of the fuzzy four-sphere requires two non-trivial steps:

- a sort of Kaluza–Klein extension of the coordinates of the classical sphere from five to fifteen coordinates;
- the extension of the gauge group from $SU(2)$ to $U(4)$, not expected from the projector point of view.

We have only an indirect check that this noncommutative solution is a smooth deformation of the classical $SU(2)$ BPST instanton; however, we believe that this is the minimal way to realize such an extension.

As a last remark, we want to comment on the possibility of defining a topological number, which extends the non-trivial second Chern class of BPST instantons. As a noncommutative deformation we suggest to choose the five-dimensional Chern–Simons term:

$$S_{\mathrm{ch}} = -\frac{5\beta}{12}\mathrm{Tr}\left[\frac{1}{5(N+2)\rho}\epsilon^{\mu\nu\lambda\rho\sigma}X_\mu X_\nu X_\lambda X_\rho X_\sigma - 4\rho^2 X_\mu X_\mu\right]. \qquad (5.17)$$

Since the solution, defined in formula (5.15), is a (reducible) representation of the fuzzy four-sphere algebra, the evaluation of S_{CS} on it reduces to the evaluation of the following simplified action

$$S_m = \beta\,\mathrm{Tr}(X_\mu X_\mu). \qquad (5.18)$$

To obtain a well-defined result, we must compare the noncommutative solution

$$\begin{aligned} S_m\left(X_i^{\mathrm{tot}}\right) &= \beta\rho^2[\mathrm{Tr}(G_iG_i)_{N+2} + \mathrm{Tr}(G_iG_i)_N] \\ &= \beta\rho^2\,\mathrm{Tr}(1)_{N+1}[N(N+1) + (N+5)(N+6)] \\ &= 2\beta\rho^2\,\mathrm{Tr}(1)_{N+1}[N^2 + 6N + 15] \end{aligned} \qquad (5.19)$$

with the background of a $U(2)$ gauge theory (and not $U(4)$)

$$\begin{aligned} S_m\left(X_i^{bg}(U(2))\right) &= 2\beta\rho^2\,\mathrm{Tr}(G_iG_i)_{N+1} \\ &= 2\beta\rho^2\,\mathrm{Tr}(1)_{N+1}(N+1)(N+5) = 2\beta\rho^2\,\mathrm{Tr}(1)_{N+1}(N^2+6N+5). \end{aligned} \qquad (5.20)$$

as it is clear from the following calculation:

$$S_m\left(X_i^{\mathrm{tot}}\right) - S_m\left(X_i^{bg}(U(2))\right) = 20\beta\rho^2\,\mathrm{Tr}(1)_{N+1}. \qquad (5.21)$$

Since all the dependence on N is contained in the term $\mathrm{Tr}(1)_{N+1}$, it is enough to renormalize the trace of the Chern–Simons term with $\mathrm{Tr}(1)_{N+1}$, to obtain an integer number, which probably coincides with the classical topological number, under the hypothesis that such limit is smooth. This analysis is however probably incomplete since there is an ambiguity of adding constant terms to equation (5.15) or to the background to reach the classical limit (see the final discussion of noncommutative monopoles in [15]). Solving such ambiguity requires a deeper mathematical comprehension of the results of this paper.

6. Conclusions

In this work, we have succeeded in showing that our method based on projective modules and matrix models can be extended to the non-Abelian case in 4d. In particular, the extension of the instantons to the fuzzy four-sphere requires two important steps:

(i) The Kaluza–Klein mechanism, which extends the five coordinates of the S^4 sphere to the fifteen ones of the noncommutative case. It should be noted that the necessary decoupling of the extra coordinates $w_{\mu\nu}$ is assured directly at the projector level.
(ii) The extension of the gauge group from $SU(2)$ to $U(2)$ for what concerns the projector and to $U(4)$ for the connection. Such distinction is a consequence of the dimensionality of the fuzzy four-sphere representations.

In the last part of this work, we suggest how to extend the second non-trivial Chen class of the instanton to the five-dimensional Chern–Simons action. We find agreement with the existence of a topological number also at a noncommutative level, although our proposal must be supported by more rigorous mathematical arguments.

Appendix

The quasi-unitary operator U defined in equations (5.1) and (5.2), when acting on the background, induces a non-trivial noncommutative topology, extension of the classical $SU(2)$ BPST instanton. In this appendix, we show how to check the main property, which is used in equations (5.7) and (5.14)

$$\left(U^\dagger X_i^{bg} U\right)_{N+1} = \left(\begin{array}{c|c|c} (G_i)_{N+2} & 0 & 0 \\ \hline 0 & (G_i)_N & 0 \\ \hline 0 & 0 & 0 \end{array}\right). \tag{A.1}$$

First we note that it is worth separating U into two pieces, $U = U_1 + U_2$, as in equation (5.10), and studying the two cases separately:

$$\begin{aligned} \left(U_1^\dagger X_i^{bg} U_1\right)_{N+1} &= \left(\begin{array}{c|c|c} (G_i)_{N+2} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \\ \left(U_2^\dagger X_i^{bg} U_2\right)_{N+1} &= \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & (G_i)_N & 0 \\ \hline 0 & 0 & 0 \end{array}\right). \end{aligned} \tag{A.2}$$

The second case is rather evident and we concentrate ourselves only on the U_1 case

$$\left(U^\dagger X_i^{bg} U\right)_{N+1} = \begin{pmatrix} U_{11} G_i U_{11}^\dagger & U_{11} G_i U_{12}^\dagger & U_{11} G_i U_{13}^\dagger & U_{11} G_i U_{14}^\dagger \\ U_{12} G_i U_{11}^\dagger & U_{12} G_i U_{12}^\dagger & U_{12} G_i U_{13}^\dagger & U_{12} G_i U_{14}^\dagger \\ U_{13} G_i U_{11}^\dagger & U_{13} G_i U_{12}^\dagger & U_{13} G_i U_{13}^\dagger & U_{13} G_i U_{14}^\dagger \\ U_{14} G_i U_{11}^\dagger & U_{14} G_i U_{12}^\dagger & U_{14} G_i U_{13}^\dagger & U_{14} G_i U_{14}^\dagger \end{pmatrix}. \tag{A.3}$$

Since this case is rather intricate, we firstly check the dimension of this representation. When restricting $U^\dagger X_i^{bg} U$ to a total fixed number $\sum n_i = N+1$, the following properties are clear:

(1) $\left(U_{11} G_i U_{11}^\dagger\right)_{N+1}$ is a square matrix, having the same dimension of the $(G_i)_{N+1}$ representation, i.e. d_{N+1};

(2) in the case $\left(U_{11} G_i U_{12}^\dagger\right)_{N+1}$, its elements $\neq 0$ are concentrated into a rectangular matrix with a short side $\frac{(N+2)(N+3)}{2}$;

(3) in the case $\left(U_{11} G_i U_{13}^\dagger\right)_{N+1}$ the short side is $(N+2)$;

(4) finally $\left(U_{11} G_i U_{14}^\dagger\right)_{N+1}$ can be reduced to a single column.

In summary, we have the following structure of the representation (A.3), (adding for completeness the term due to U_2):

$$
\left(\begin{array}{c||c|c||c|c||c|c}
U_{11}G_iU_{11}^\dagger & U_{11}G_iU_{12}^\dagger & 0 & U_{11}G_iU_{13}^\dagger & 0 & U_{11}G_iU_{14}^\dagger & 0 \\ \hline
U_{12}G_iU_{11}^\dagger & U_{12}G_iU_{12}^\dagger & 0 & U_{12}G_iU_{13}^\dagger & 0 & U_{12}G_iU_{14}^\dagger & 0 \\ \hline
0 & 0 & (G_i)_N & 0 & 0 & 0 & 0 \\ \hline
U_{13}G_iU_{11}^\dagger & U_{13}G_iU_{12}^\dagger & 0 & U_{13}G_iU_{13}^\dagger & 0 & U_{13}G_iU_{14}^\dagger & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
U_{14}G_iU_{11}^\dagger & U_{14}G_iU_{12}^\dagger & 0 & U_{14}G_iU_{13}^\dagger & 0 & U_{14}G_iU_{14}^\dagger & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right). \tag{A.4}
$$

If we ignore all the columns and rows which are trivially null and paste only those strips in which it is possible to find entries $\neq 0$, as discussed before, we end up with a new square matrix of dimension

$$
\begin{aligned}
d &= \frac{(N+2)(N+3)(N+4)}{6} + \frac{(N+2)(N+3)}{2} + (N+2) + 1 \\
&= \frac{(N+3)(N+4)(N+5)}{6}
\end{aligned} \tag{A.5}
$$

which is the dimension of the irreducible representation $(G_i)_{N+2}$. Verifying that we obtain exactly the same values of the representation $(G_i)_{N+2}$ is rather tedious. Let us briefly check the case of G_5, whose representation is a diagonal matrix,

$$
G_5 = \sum_{n_1,n_2,n_3,n_4=0}^{\infty} (n_1 + n_2 - n_3 - n_4)|n_1, n_2, n_3, n_4\rangle\langle n_1, n_2, n_3, n_4| \tag{A.6}
$$

and therefore we must control only the diagonal terms:

$$
\begin{aligned}
U_{11}G_5U_{11}^\dagger &= \sum_{n_1,n_2,n_3,n_4=0}^{\infty} (n_1 + n_2 + 1 - n_3 - n_4)|n_1, n_2, n_3, n_4\rangle\langle n_1, n_2, n_3, n_4| \\
U_{12}G_5U_{12}^\dagger &= \sum_{n_1,n_2,n_3,n_4=0}^{\infty} (n_2 + 1 - n_3 - n_4)|n_2, n_3, n_4, 0\rangle\langle n_2, n_3, n_4, 0| \\
U_{13}G_5U_{13}^\dagger &= \sum_{n_1,n_2,n_3,n_4=0}^{\infty} (-n_3 - n_4 - 1)|n_3, n_4, 0, 0\rangle\langle n_3, n_4, 0, 0| \\
U_{14}G_5U_{14}^\dagger &= \sum_{n_1,n_2,n_3,n_4=0}^{\infty} (-n_4 - 1)|n_4, 0, 0, 0, 0\rangle\langle n_4, 0, 0, 0|.
\end{aligned} \tag{A.7}
$$

Let us first consider the last term which is the simplest to discuss. In fact restricting it to a fixed total oscillator number $(N+1)$, there is only one element left

$$
\left(U_{14}G_5U_{14}^\dagger\right)_{N+1} = -(N+2)|N+1, 0, 0, 0\rangle\langle N+1, 0, 0, 0| \tag{A.8}
$$

whose value $-(N+2)$ corresponds to the minimum value of the representation $(G_5)_{N+2}$. Studying such representation in detail, we note that this minimum term is repeated $(N+3)$ times, so we must find other $(N+2)$ copies of it. It is not difficult to realize that all those terms come from the restriction of

$$
\left(U_{13}G_5U_{13}^\dagger\right)_{N+1} = -(N+2) \sum_{n_3,n_4}^{n_3+n_4=N+1} |n_3, n_4, 0, 0\rangle\langle n_3, n_4, 0, 0|. \tag{A.9}
$$

Proceeding this way, we can verify that the terms $(U_{1i}G_5U_{1i}^\dagger)_{N+1}$ $(i = 1, 2)$ never reach the minimum value $-(N+2)$ and it instead contribute to complete the representation $(G_5)_{N+2}$. With some patience, the cases $(G_{1\pm})_{N+2}$ and $(G_{2\pm})_{N+2}$ can be successfully checked.

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